Group stability and Property (T)

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In recent years, there has been a considerable amount of interest in the stability of a finitely-generated group Γ with respect to a sequence of groups $\{G_n\}_{n=1}^{\infty}$, equipped with bi-invariant metrics $\{d_n\}_{n=1}^{\infty}$. We consider the case $G_n = \mathrm{U}(n)$ (resp. $G_n = \mathrm{Sym}(n)$), equipped with the normalized Hilbert-Schmidt metric d_n^{HS} (resp. the normalized Hamming metric d_n^{Hamming}). Our main result is that if Γ is infinite, hyperlinear (resp. sofic) and has Property (T), then it is not stable with respect to $(\mathrm{U}(n), d_n^{\mathrm{HS}})$ (resp. $(\mathrm{Sym}(n), d_n^{\mathrm{Hamming}})$).

This answers a question of Hadwin and Shulman regarding the stability of $\mathrm{SL}_3(\mathbb{Z})$. We also deduce that the mapping class group $\mathrm{MCG}(g)$, $g \geq 3$, and $\mathrm{Aut}(\mathbb{F}_n)$, $n \geq 3$, are not stable with respect to $(\mathrm{Sym}(n), d_n^{\mathrm{Hamming}})$.

Our main result exhibits a difference between stability with respect to the normalized Hilbert-Schmidt metric on $\mathrm{U}(n)$ and the (unnormalized) p-Schatten metrics, since many groups with Property (T) are stable with respect to the latter metrics, as shown by De Chiffre-Glebsky-Lubotzky-Thom and Lubotzky-Oppenheim.

We suggest a more flexible notion of stability that may repair this deficiency of stability with respect to $(\mathrm{U}(n), d_n^{\mathrm{HS}})$ and $(\mathrm{Sym}(n), d_n^{\mathrm{Hamming}})$.

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1. Introduction

Let $\mathcal{G} = \{(G_n, d_n)\}_{n=1}^{\infty}$ be a family of groups G_n endowed with bi-invariant metrics d_n, i.e., $d_n(ag_1b, ag_2b) = d_n(g_1, g_2)$ for all $g_1, g_2, a, b \in G_n$. Here are some examples:

(1) $\mathcal{P} = \left\{ \left(\mathrm{Sym}(n), d_n^{\mathrm{Hamming}} \right) \right\}_{n=1}^{\infty}$, where d_n^{Hamming} is the normalized Hamming metric on $\mathrm{Sym}(n)$: for $\sigma, \tau \in \mathrm{Sym}(n)$,

$$d_n^{\mathrm{Hamming}}(\sigma, \tau) = \frac{1}{n} \cdot |\{x \in [n] \mid \sigma(x) \neq \tau(x)\}| \ ,$$

where $[n] = \{1, \ldots, n\}$.
(2) $\mathrm{HS} = \left\{ \left(\mathrm{U}(n), d_n^{\mathrm{HS}} \right) \right\}_{n=1}^{\infty}$, where d_n^{HS} is the normalized Hilbert-Schmidt metric: for $A, B \in \mathrm{U}(n)$,

$$d_n^{\mathrm{HS}}(A, B) = \|A - B\|_{\mathrm{HS}} \ , \qquad \text{where } \|T\|_{\mathrm{HS}} = \left(\mathrm{Tr}\left(\frac{1}{n} \cdot T^*T \right) \right)^{1/2} \ .$$

(3) $\mathcal{G}^{(p)} = \left\{ \left(\mathrm{U}(n), d_n^{(p)} \right) \right\}_{n=1}^{\infty}$, for any fixed $1 \leq p < \infty$, where $d_n^{(p)}$ is the Schatten p-norm (see [8], Section IV.2): for $A, B \in \mathrm{U}(n)$,

$$d_n^{(p)}(A, B) = \|A - B\|_p \ , \qquad \text{where } \|T\|_p = \left(\mathrm{Tr}\left((T^*T)^{p/2} \right) \right)^{1/p} \ .$$

The case $p = 2$ is of special interest: this is the standard L^2-norm, a.k.a. the Frobenius norm, denoted $\| \cdot \|_{\mathrm{F}}$. Note that $d_n^{(2)} = \sqrt{n} \cdot d_n^{\mathrm{HS}}$. The proofs in Section 2 also make use of the Frobenius norm $\| \cdot \|_{\mathrm{F}}$ for non-square matrices, which is defined by the same formula: $\|T\|_{\mathrm{F}} = \mathrm{Tr}(T^*T)^{1/2}$.
(4) $\mathcal{G}^{(\infty)} = \left\{ \left(\mathrm{U}(n), d_n^{(\infty)} \right) \right\}_{n=1}^{\infty}$, where the metric $d_n^{(\infty)}$ is defined for $A, B \in \mathrm{U}(n)$ by $d_n^{(\infty)}(A, B) = \|A - B\|_\infty$, where $\| \cdot \|_\infty$ is the operator norm.

Let $\mathbb{F}$ be a free group on a finite set S. Let Γ be a quotient of $\mathbb{F}$, and denote the quotient map by $\pi : \mathbb{F} \to \Gamma$. The objects S, $\mathbb{F}$, Γ and π shall remain fixed throughout this paper.

From now on, for a group G, a function $f : S \to G$ and an element $w \in \mathbb{F}$, we write $f(w)$ for the element of G resulting from applying the substitution $s \mapsto f(s)$ to the word w.

Definition 1.1.

i) A $\mathcal{G}$-*stability-challenge* for Γ is a sequence $(f_k)_{k=1}^{\infty}$ of functions $f_k : S \to G_{n_k}$, $n_k \in \mathbb{N}$, such that for every $w \in \mathrm{Ker}(\pi)$,

$$d_{n_k}\left(f_k(w), \mathbf{1}_{G_{n_k}} \right) \xrightarrow{k \to \infty} 0 \ .$$

ii) Let $(f_k)_{k=1}^{\infty}$ be a $\mathcal{G}$-stability-challenge for Γ. A *solution* for $(f_k)_{k=1}^{\infty}$ is a sequence of functions $(g_k)_{k=1}^{\infty}$, $g_k : S \to G_{n_k}$, such that for every $w \in \mathrm{Ker}\,(\pi)$, $g_k\,(w) = \mathbf{1}_{G_{n_k}}$ (i.e., g_k defines a homomorphism $\Gamma \to G_{n_k}$), and

$$\sum_{s \in S} d_{n_k}\,(f_k\,(s)\,, g_k\,(s)) \overset{k \to \infty}{\longrightarrow} 0\,.$$

iii) The group Γ is $\mathcal{G}$-*stable* if every $\mathcal{G}$-stability-challenge for Γ has a solution.

While the above definition of a $\mathcal{G}$-stable group made use of a given presentation of Γ as a quotient of a free group, it is in fact a group property, independent of the specific presentation (cf. [1]).

In recent years, there has been a considerable amount of interest in "group stability" (see [1], [35], [4] and the references therein). One of the main motivations is the study of $\mathcal{G}$-approximations of Γ:

Definition 1.2. For $\mathcal{G}$ and Γ as above, we say that Γ is $\mathcal{G}$-*approximated* if there is a sequence of integers $\{n_k\}_{k=1}^{\infty}$, and a sequence $(\varphi_k)_{k=1}^{\infty}$ of functions $\varphi_k : \Gamma \to G_{n_k}$, such that

$$\forall g, h \in \Gamma \quad \lim_{k \to \infty} d_{n_k}\,(\varphi_k\,(gh)\,, \varphi_k\,(g)\,\varphi_k\,(h)) = 0$$

and

$$\forall 1_{\Gamma} \neq g \in \Gamma \quad \limsup_k d_{n_k}\left(\varphi_k\,(g)\,, 1_{G_{n_k}}\right) > 0$$

In classical terminology, $\mathcal{P}$-approximated groups (for $\mathcal{P}$ as in Example (1) above) are called *sofic* groups, and HS-approximated groups are called *hyperlinear* groups (or *Connes embeddable* [35]). It is a well-known open problem, due to Gromov (resp. Connes), whether every group is sofic (resp. hyperlinear). Note that all sofic groups are hyperlinear.

In [10], it was shown for the first time that there are finitely presented groups Γ which are not $\left(\mathrm{U}\,(n), d_n^{(2)}\right)$-approximated (i.e., *Frobenius-approximated*), and this result was extended in [25] to all $1 < p < \infty$. The groups Γ in those papers are finite central extensions of suitable lattices $\bar{\Gamma}$ in simple Lie groups of rank $r \geq 3$ over local non-archimedean fields. The key point there is that these groups Γ and $\bar{\Gamma}$ are $\mathcal{G}^{(2)}$-stable (and even $\mathcal{G}^{(p)}$-stable). This is proved as a corollary to the vanishing result $H^i\,(\Gamma, V) = 0$ for every $i = 1, \ldots, r - 1$ and for all actions of Γ on Hilbert spaces V (and the same for many Banach spaces). The case $i = 2$ gives the stability of Γ. Vanishing for $i = 1$ is equivalent to Γ having Property (T), so all the groups treated there have Kazhdan's Property (T).

The goal of the present paper is to show that these examples are neither $\mathcal{P}$-stable nor HS-stable. In fact, we prove a much more general result, which is of independent interest:

Theorem 1.3.

i) *If Γ is sofic and has Property* (T)*, then it is not $\mathcal{P}$-stable, unless it is finite.*
ii) *If Γ is hyperlinear and has Property* (T)*, then it is not* HS-*stable, unless it is finite.*

As every finitely generated linear group is sofic, all lattices in higher rank simple algebraic groups over local fields are neither $\mathcal{P}$-stable nor HS-stable. In particular, this holds for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, answering a question of Hadwin and Shulman [13].

Theorem 1.3 is a corollary of the following:

Theorem 1.4. *Assume that Γ has Kazhdan's Property* (T)*, and is either $\mathcal{P}$-stable or* HS-*stable. Then, Γ has only finitely many finite-index subgroups.*

Theorem 1.4 is proved in Section 2. We can already show how it implies Theorem 1.3:

Proof of Theorem 1.3. Assume that Γ is sofic and $\mathcal{P}$-stable. A well-known observation (see Theorem 2 in [11]) says that, in this case, Γ is residually-finite. If further, Γ has Property (T), then by Theorem 1.4, it has only finitely many finite-index subgroups, and so it is finite.

Assume, instead, that Γ is hyperlinear and HS-stable. It is easy to see that in this case too, Γ is residually-finite. Indeed, arguing as in [11], we see that Γ is residually-linear, and so it is residually-finite since finitely-generated linear groups are residually-finite. So, as before, if, further, Γ has Property (T), it must be finite. □

In Section 3, we deduce the following:

Theorem 1.5. *For $g \geq 3$, the mapping class group $\mathrm{MCG}(g)$ of an orientable closed surface of genus g is not $\mathcal{P}$-stable. For $n \geq 3$, the (outer) automorphism group $\mathrm{Aut}(\mathbb{F}_n)$ (and $\mathrm{Out}(\mathbb{F}_n)$) is not $\mathcal{P}$-stable.*

We do not know if the theorem also holds for HS-stability. Its proof uses Theorem 1.3 and Proposition 3.2, which states that if Γ is $\mathcal{P}$-stable then so is its quotient by a normal subgroup N, provided that N is a finitely-generated group.

In Section 4, we discuss variations of Property (T) (e.g., Property (τ) and relative Property (T)), the stability of semidirect products and free products with amalgamation and a flexible notion of stability, and suggest problems for further research.

In the appendix we reprove Theorem 1.4 using the representation theory of Chevalley groups, under the assumption that Γ has an infinite linear quotient. In fact, we prove a somewhat stronger version of the theorem for such groups Γ.

2. The proof of Theorem 1.4

Before we begin, we record a simple observation regarding $\mathcal{P}$-stability. Fix formal elements $\{v_i\}_{i=1}^{\infty}$, and for every $n \in \mathbb{N}$, let $\mathcal{B}_n = (v_1, \ldots, v_n)$ serve as an ordered basis for a complex vector space $\mathcal{H}_n$. A permutation $\sigma \in \mathrm{Sym}\,(n) \cong \mathrm{Sym}\,(\mathcal{B}_n)$ extends uniquely to an element of $\mathrm{U}\,(\mathcal{H}_n) \cong \mathrm{U}\,(n)$, giving an embedding $\iota : \mathrm{Sym}\,(n) \to \mathrm{U}\,(n)$. For permutations $\sigma, \tau \in \mathrm{Sym}\,(n)$,

$$d_n^{\mathrm{HS}}\left(\iota\left(\sigma\right), \iota\left(\tau\right)\right) = \sqrt{2 \cdot d_n^{\mathrm{Hamming}}\left(\sigma, \tau\right)} \ .$$

Therefore, for $\mathcal{G}_0 = \left\{\left(\iota\left(\mathrm{Sym}\,(n)\right), d_n^{\mathrm{HS}}\right)\right\}_{n=1}^{\infty}$, the group Γ is $\mathcal{P}$-stable if and only if it is $\mathcal{G}_0$-stable. We also refer to $\mathcal{G}_0$-stability-challenges and $\mathcal{P}$-stability challenges interchangeably.

For a linear operator $T : \mathcal{H} \to \mathcal{H}$ on a finite-dimensional vector space $\mathcal{H}$ and a basis $\mathcal{B}$ for $\mathcal{H}$, we write $[T]_{\mathcal{B}}$ for the matrix representing T with respect to $\mathcal{B}$.

Lemma 2.1. *Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space with orthonormal ordered basis $\mathcal{B} = (v_1, \ldots, v_n)$. Let $\mathcal{B}_0 = (v_1, \ldots, v_{n-1})$ and $\mathcal{H}_0 = \mathrm{span}_{\mathbb{C}}\, \mathcal{B}_0$. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Write $P_0 : \mathcal{H} \to \mathcal{H}_0$ for the orthogonal projection onto $\mathcal{H}_0$, and $T_0 = P_0 \circ T \mid_{\mathcal{H}_0}: \mathcal{H}_0 \to \mathcal{H}_0$. Then:*

i) *If T permutes $\mathcal{B}$, then there is a linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, which permutes $\mathcal{B}_0$, such that*

$$\|T_0 - A_0\|_{\mathrm{F}} \leq 1$$

iii) *If T is unitary, then there is a unitary linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, such that*

$$\|T_0 - A_0\|_{\mathrm{F}} \leq 1$$

In both cases, the inclusion map $f : \mathcal{H}_0 \to \mathcal{H}$ satisfies

$$\|T^{-1} \circ f \circ A_0 - f\|_{\mathrm{F}} \leq 2$$

Proof. (i) Assume that T permutes $\mathcal{B}$. Denote $T^{-1}\left(v_n\right) = v_{i_0}$, $1 \leq i_0 \leq n$. Define a linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$ on the elements of the basis $\mathcal{B}_0$ by

$$A_0\left(v_i\right) = \begin{cases} T\left(v_i\right) & i \neq i_0 \\ T\left(v_n\right) = T\left(T\left(v_i\right)\right) & i = i_0 \end{cases} \ .$$

Then, only the i_0-th column of $[T_0 - A_0]_{\mathcal{B}_0}$ may be nonzero, and its norm is 0 if $i_0 = n$, or 1 otherwise. In any case,

$$\|T_0 - A_0\|_F \le 1$$

(ii) Assume that T is unitary. Take a polar decomposition $T_0 = A_0 \cdot \sqrt{T_0^* T_0}$ of T_0, where $A_0 \in U(\mathcal{H}_0)$ (see Theorem 3.1.9(c) in [15]). Note that, generally, A_0 is only guaranteed to exist, but is not unique. Then,

$$\|T_0 - A_0\|_F = \|A_0^{-1} \cdot (T_0 - A_0)\|_F = \|\sqrt{T_0^* T_0} - I_{n-1}\|_F \tag{2.1}$$

Let $u \in M_{1 \times (n-1)}(\mathbb{C})$ be the bottom row of $[T]_{\mathcal{B}}$, with the rightmost entry removed. Since T is unitary, we have $\|u\| \le 1$. Partition $[T]_{\mathcal{B}}$ as a block matrix, where $[T_0]_{\mathcal{B}_0}$ is the top-left block, and u is the bottom-left block. Since $T^* T = I$, we get $[T_0^* T_0]_{\mathcal{B}_0} + u^* u = I_{n-1}$, i.e., $[T_0^* T_0]_{\mathcal{B}_0} = I_{n-1} - u^* u$. The eigenvalues of $u^* u$ are 0 (with multiplicity $n-2$), and $\langle u, u \rangle = \|u\|^2$ (with multiplicity 1, corresponding to the right eigenvector u^*). So, $\sqrt{T_0^* T_0}$ is a unitarily diagonalizable operator whose eigenvalues are 1, with multiplicity $n-2$, and $\sqrt{1 - \|u\|^2}$, with multiplicity 1. Hence,

$$\|\sqrt{T_0^* T_0} - I_{n-1}\|_F = \left| \sqrt{1 - \|u\|^2} - 1 \right| \le 1$$

which, together with (2.1), implies the desired result.

As for the last claim,

$$
\begin{aligned}
\|T^{-1} \circ f \circ A_0 - f\|_F &= \|f \circ A_0 - T \circ f\|_F \\
&= \|f \circ A_0 - T \mid_{\mathcal{H}_0}\|_F \\
&\le \|f \circ (A_0 - T_0)\|_F + \|f \circ T_0 - T \mid_{\mathcal{H}_0}\|_F \\
&= \|T_0 - A_0\|_F + \|[(f \circ P_0 \circ T - T) \mid_{\mathcal{H}_0}]_{\mathcal{B}_0}^{\mathcal{B}}\|_F \\
&= \|T_0 - A_0\|_F + \|u\|_F \\
&\le 1 + 1 = 2
\end{aligned}
$$

where u is, again, the bottom-left row of $[T]_{\mathcal{B}}$, with the rightmost entry removed. □

Lemma 2.2. *Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. Let $U_1, \ldots, U_l \in U(\mathcal{H})$ and $E_1, \ldots, E_l \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$. Let $c \ge 0$, and assume that $\|E_i\|_F \le c$ for all $1 \le i \le l$. Then,*

$$\|\prod_{i=1}^{l} (U_i + E_i) - \prod_{i=1}^{l} U_i\|_F \le (c+1)^l$$

Proof. Let $\varnothing \ne A_0 \subset [l]$ (where $[l] = \{1, \ldots, l\}$). For each $1 \le i \le l$, denote $M_i = \begin{cases} E_i & i \in A_0 \\ U_i & i \notin A_0 \end{cases}$. Let $1 \le k \le l$, and consider the product $\prod_{i=1}^{k} M_i$. On one hand, if $k \notin A_0$, then

$$\|\prod_{i=1}^{k} M_i\|_{\mathrm{F}} = \|\left(\prod_{i=1}^{k-1} M_i\right) \cdot U_k\|_{\mathrm{F}} = \|\left(\prod_{i=1}^{k-1} M_i\right)\|_{\mathrm{F}}$$

since the Frobenius norm $\|\cdot\|_{\mathrm{F}}$ is invariant under multiplication by unitary operators. On the other hand, if $k \in A_0$, then

$$\|\prod_{i=1}^{k} M_i\|_{\mathrm{F}} = \|\left(\prod_{i=1}^{k-1} M_i\right) \cdot E_k\|_{\mathrm{F}} \leq \|\prod_{i=1}^{k-1} M_i\|_{\mathrm{F}} \cdot \|E_k\|_{\mathrm{F}}$$

since $\|\cdot\|_{\mathrm{F}}$ is submultiplicative. So, we conclude by induction that

$$\|\prod_{i=1}^{k} M_i\|_{\mathrm{F}} \leq \prod_{i \in A_0} \|E_i\|_{\mathrm{F}} \leq c^{|A_0|} .$$

Together with the triangle inequality, this implies that

$$\|\prod_{i=1}^{l} (U_i + E_i) - \prod_{i=1}^{l} U_i\|_{\mathrm{F}} \leq \sum_{\varnothing \neq A \subset [l]} c^{|A|}$$

$$= \sum_{i=1}^{l} \binom{l}{i} \cdot c^i$$

$$\leq (c+1)^l . \qquad \square$$

For a word $w \in \mathbb{F}$, write $|w|$ for the *length* of w, i.e., the length of w when written as a reduced word over $S^{\pm}$. Recall that we write π for the fixed quotient map $\pi : \mathbb{F} \to \Gamma$.

Proposition 2.3. *Let $(\mathcal{H}, \alpha)$ be a finite-dimensional unitary representation of Γ. Let $\mathcal{H}_0 \subset \mathcal{H}$ be a subspace of co-dimension 1. Let $\mathcal{B}_0 \subset \mathcal{B}$ be orthonormal bases for $\mathcal{H}_0, \mathcal{H}$, respectively. Then:*

  i) *There is a function $\rho : S \to \mathrm{U}(\mathcal{H}_0)$, such that $\|\rho(w) - I\|_{\mathrm{F}} \leq 3^{|w|}$ for every $w \in \mathrm{Ker}(\pi)$, and the inclusion map $f : \mathcal{H}_0 \to \mathcal{H}$ satisfies*

$$\|\alpha(s^{-1}) \circ f \circ \rho(s) - f\|_{\mathrm{F}} \leq 2$$

  for each $s \in S$.

  ii) *If, furthermore, each $\alpha(s)$ permutes $\mathcal{B}$, then ρ above can be chosen such that each $\rho(s)$ permutes $\mathcal{B}_0$.*

Proof. Define $\alpha_0 : S \to \mathrm{End}_{\mathbb{C}} \mathcal{H}$ by $\alpha_0(s) = P_0 \circ \alpha(s)|_{\mathcal{H}_0}$, where $P_0 : \mathcal{H} \to \mathcal{H}_0$ is the orthogonal projection. By Lemma 2.1, applied to $\alpha(s)$ for each $s \in S$ separately, there is a function $\rho : S \to \mathrm{U}(\mathcal{H}_0)$, such that

$$\|\alpha_0(s) - \rho(s)\|_{\mathrm{F}} \le 1 \tag{2.2}$$

and if each $\alpha(s)$ permutes $\mathcal{B}$, then ρ can be chosen so that each $\rho(s)$ permutes $\mathcal{B}_0$. In any case, Lemma 2.1 guarantees that

$$\|\alpha(s^{-1}) \circ f \circ \rho(s) - f\|_{\mathrm{F}} \le 2$$

for each $s \in S$. Let $s \in S$. By considering the matrix representations of $\rho(s) \oplus \mathbf{1}_{\mathcal{H}_0^\perp}$ and $\alpha(s)$, and using the fact that $\alpha(s)$ is unitary, we see that

$$\|\rho(s) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(s)\|_{\mathrm{F}}^2 \le \|\rho(s) - \alpha_0(s)\|_{\mathrm{F}}^2 + 3 .$$

Hence, from (2.2), we conclude that

$$\|\rho(s) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(s)\|_{\mathrm{F}} \le \left(\|\rho(s) - \alpha_0(s)\|_{\mathrm{F}}^2 + 3\right)^{1/2} \le 2$$

We would like to bound $\|\rho(\cdot) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(\cdot)\|_{\mathrm{F}}$, evaluated at a word $w \in \mathbb{F}$, and so we also need to bound $\|\rho(s)^{-1} \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(s)^{-1}\|_{\mathrm{F}}$. But, in general $\|A^{-1} - B^{-1}\|_{\mathrm{F}} = \|A - B\|_{\mathrm{F}}$ for $A, B \in \mathrm{U}(\mathcal{H})$, and so

$$\|\rho(s)^{-1} \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(s^{-1})\|_{\mathrm{F}} \le 2 .$$

Let $w \in \mathbb{F}$. By Lemma 2.2, the above implies that

$$\|\rho(w) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(w)\|_{\mathrm{F}} \le (2+1)^{|w|} = 3^{|w|} .$$

Assume further that $w \in \mathrm{Ker}(\pi)$. Then $\alpha(w) = I$, and so,

$$\begin{aligned}
\|\rho(w) - I\|_{\mathrm{F}} &= \|\rho(w) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - I\|_{\mathrm{F}} \\
&\le \|\rho(w) \oplus \mathbf{1}_{\mathcal{H}_0^\perp} - \alpha(w)\|_{\mathrm{F}} + \|\alpha(w) - I\|_{\mathrm{F}} \\
&\le 3^{|w|} \qquad \square
\end{aligned}$$

Henceforth, given representations $\mathcal{H}_1$ and $\mathcal{H}_2$ of Γ, we treat $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ as a Γ-representation with the action given by $g \cdot f = g \circ f \circ g^{-1}$ for $g \in \Gamma$ and $f \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$.

Proposition 2.4. *Let $\mathcal{H}_0 \subsetneq \mathcal{H}$ be finite-dimensional complex Hilbert spaces, and write $f : \mathcal{H}_0 \to \mathcal{H}$ for the inclusion map. Let $\alpha : \Gamma \to \mathrm{U}(\mathcal{H})$ and $\beta : \Gamma \to \mathrm{U}(\mathcal{H}_0)$ be unitary representations. Then:*

i) *If $(\mathcal{H}, \alpha)$ is irreducible, then $\|f - h\|_{\mathrm{F}} = \|f\|_{\mathrm{F}}$ for every morphism of Γ-representations $h : \mathcal{H}_0 \to \mathcal{H}$.*

ii) If $\mathcal{B}_0 \subset \mathcal{B}$ are orthonormal bases for $\mathcal{H}_0, \mathcal{H}$, respectively, each $\beta(s)$ permutes $\mathcal{B}_0$, each $\alpha(s)$ permutes $\mathcal{B}$, and the action $\Gamma \curvearrowright^\alpha \mathcal{B}$ of Γ on $\mathcal{B}$ through α is transitive, then $\|f - h\|_{\mathrm{F}} \geq \frac{1}{\sqrt{2}} \cdot \|f\|_{\mathrm{F}}$ for every morphism of Γ-representations $h : \mathcal{H}_0 \to \mathcal{H}$.

Proof. (i) Since $\dim_{\mathbb{C}} \mathcal{H}_0 < \dim_{\mathbb{C}} \mathcal{H}$ and $(\mathcal{H}, \alpha)$ is irreducible, Schur's Lemma implies the only morphism of representations $\mathcal{H}_0 \to \mathcal{H}$ is the zero morphism, and so the result follows.

(ii) For $b_0 \in \mathcal{B}_0$ and $b \in \mathcal{B}$, let $E_{b_0, b} : \mathcal{H}_0 \to \mathcal{H}$ be the linear map sending $b_0 \mapsto b$, and sending every other element of $\mathcal{B}_0$ to zero. Then, $\{E_{b_0, b}\}_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}}$ is a basis for $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_0, \mathcal{H})$. The inner product for which $\{E_{b_0, b}\}_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}}$ is an orthonormal basis makes the Γ-representation $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_0, \mathcal{H})$ unitary. The group Γ acts on $\mathcal{B}_0 \times \mathcal{B}$ by $\gamma \cdot (b_0, b) = (\gamma \cdot b_0, \gamma \cdot b)$. A map $T \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_0, \mathcal{H})$, represented as $T = \sum_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}} \lambda_{b_0, b} \cdot E_{b_0, b}$, is a morphism of representations if and only if the mapping $(b_0, b) \mapsto \lambda_{b_0, b}$ is constant on each Γ-orbit of $\mathcal{B}_0 \times \mathcal{B}$.

Let $\mathcal{O}_1^{\mathcal{B}_0}, \dots, \mathcal{O}_k^{\mathcal{B}_0}$ be the orbits of the action $\Gamma \curvearrowright \mathcal{B}_0$. Take an orbit $\mathcal{O}$ of the action $\Gamma \curvearrowright \mathcal{B}_0 \times \mathcal{B}$. Then, there is a unique $1 \leq i \leq k$ such that $\mathcal{O}$ is contained in $\mathcal{O}_i^{\mathcal{B}_0} \times \mathcal{B}$. We claim that $|\mathcal{O}| \geq 2 \cdot \left|\mathcal{O}_i^{\mathcal{B}_0}\right|$. Indeed, let $(b_0, b) \in \mathcal{O}$. Then,

$$\mathrm{Stab}_\Gamma((b_0, b)) = \mathrm{Stab}_\Gamma(b_0) \cap \mathrm{Stab}_\Gamma(b) \leq \mathrm{Stab}_\Gamma(b_0) \ . \tag{2.3}$$

The action $\Gamma \curvearrowright^\alpha \mathcal{B}$ is transitive, and so $|\mathcal{B}| = [\Gamma : \mathrm{Stab}_\Gamma(b)]$. Thus,

$$[\Gamma : \mathrm{Stab}_\Gamma(b_0)] \leq |\mathcal{B}_0| < |\mathcal{B}| = [\Gamma : \mathrm{Stab}_\Gamma(b)] \ .$$

In particular, $\mathrm{Stab}_\Gamma(b_0)$ is not a subgroup of $\mathrm{Stab}_\Gamma(b)$, and so the inclusion in (2.3) is strict. Hence,

$$\begin{aligned}
|\mathcal{O}| &= [\Gamma : \mathrm{Stab}_\Gamma((b_0, b))] \\
&= [\Gamma : \mathrm{Stab}_\Gamma(b_0)] \cdot [\mathrm{Stab}_\Gamma(b_0) : \mathrm{Stab}_\Gamma((b_0, b))] \\
&\geq 2 \cdot [\Gamma : \mathrm{Stab}_\Gamma(b_0)] \\
&= 2 \cdot \left|\mathcal{O}_i^{\mathcal{B}_0}\right| \ ,
\end{aligned} \tag{2.4}$$

as claimed.

For each Γ-orbit $\mathcal{O}$ of $\mathcal{B}_0 \times \mathcal{B}$, let $c(\mathcal{O})$ be the number of elements $(b_0, b) \in \mathcal{O}$ for which $f(b_0) = b$, i.e., $c(\mathcal{O}) = |\mathcal{O} \cap \{(b_0, b_0) \mid b_0 \in \mathcal{B}_0\}|$. Then, $\sum_{\mathcal{O}} c(\mathcal{O}) = |\mathcal{B}_0|$, and for each $1 \leq i \leq k$, $\sum_{\mathcal{O} \subset \mathcal{O}_i^{\mathcal{B}_0} \times \mathcal{B}} c(\mathcal{O}) = \left|\mathcal{O}_i^{\mathcal{B}_0}\right|$. Let $h : \mathcal{H}_0 \to \mathcal{H}$ be the result of applying the orthogonal projection $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_0, \mathcal{H}) \longrightarrow \mathrm{Hom}_{\mathbb{C}\Gamma}(\mathcal{H}_0, \mathcal{H})$ to the given inclusion map $f : \mathcal{H}_0 \to \mathcal{H}$. Then, h is the morphism of representations which is closest to f under $\|\cdot\|_{\mathrm{F}}$. The $\{E_{b_0, b}\}_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}}$-coefficients of h, which are constant on each Γ-orbit of $\mathcal{B}_0 \times \mathcal{B}$, are obtained by taking the average of the coefficients of f in each Γ-orbit separately. Write $f = \sum_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}} \lambda_{b_0, b} \cdot E_{b_0, b}$ and $h = \sum_{(b_0, b) \in \mathcal{B}_0 \times \mathcal{B}} \mu_{b_0, b} \cdot E_{b_0, b}$. Then, for each Γ-orbit

$\mathcal{O}$ of $\mathcal{B}_0 \times \mathcal{B}$, the map $\lambda : \mathcal{O} \to \mathbb{C}$, defined by $\lambda (b_0, b) = \lambda_{b_0, b}$, has, in its image, $c (\mathcal{O})$ 1-s and $(|\mathcal{O}| - c (\mathcal{O}))$ 0-s, while the map $\mu : \mathcal{O} \to \mathbb{C}$, defined by $\mu (b_0, b) = \mu_{b_0, b}$, is constant, mapping all elements to $\frac{c(\mathcal{O})}{|\mathcal{O}|}$. So, writing $\sum_{\mathcal{O}}$ for a sum that runs over all Γ-orbits $\mathcal{O}$ of $\mathcal{B}_0 \times \mathcal{B}$, we deduce that

$$\|f - h\|_{\mathrm{F}}^2 = \sum_{\mathcal{O}} \left(c (\mathcal{O}) \cdot \left(1 - \frac{c (\mathcal{O})}{|\mathcal{O}|} \right)^2 + (|\mathcal{O}| - c (\mathcal{O})) \cdot \left(0 - \frac{c (\mathcal{O})}{|\mathcal{O}|} \right)^2 \right)$$

$$= \sum_{\mathcal{O}} c (\mathcal{O}) - \sum_{\mathcal{O}} \frac{c (\mathcal{O})^2}{|\mathcal{O}|} = |\mathcal{B}_0| - \sum_{\mathcal{O}} \frac{c (\mathcal{O})^2}{|\mathcal{O}|}$$

But, using Inequality (2.4), we deduce that

$$\sum_{\mathcal{O}} \frac{c (\mathcal{O})^2}{|\mathcal{O}|} = \sum_{i=1}^{k} \sum_{\mathcal{O} \subset \mathcal{O}_i^{\mathcal{B}_0} \times \mathcal{B}} \frac{c (\mathcal{O})^2}{|\mathcal{O}|} \leq \sum_{i=1}^{k} \sum_{\mathcal{O} \subset \mathcal{O}_i^{\mathcal{B}_0} \times \mathcal{B}} \frac{c (\mathcal{O})^2}{2 \cdot \left| \mathcal{O}_i^{\mathcal{B}_0} \right|}$$

$$\leq \frac{1}{2} \cdot \sum_{i=1}^{k} \frac{1}{\left| \mathcal{O}_i^{\mathcal{B}_0} \right|} \cdot \left(\sum_{\mathcal{O} \subset \mathcal{O}_i^{\mathcal{B}_0} \times \mathcal{B}} c (\mathcal{O}) \right)^2$$

$$= \frac{1}{2} \cdot \sum_{i=1}^{k} \left| \mathcal{O}_i^{\mathcal{B}_0} \right| = \frac{1}{2} \cdot |\mathcal{B}_0|$$

Thus,

$$\|f - h\|_{\mathrm{F}}^2 \geq \frac{1}{2} \cdot |\mathcal{B}_0| = \frac{1}{2} \cdot \|f\|_{\mathrm{F}}^2 \ ,$$

and so, taking square roots finishes the proof. $\square$

We recall the definition of Kazhdan's Property (T) (see Section 1.1 of [6]). Let $Q \subset \Gamma$ and $\kappa > 0$. Recall that for a unitary representation $(\mathcal{H}, \rho)$ of Γ and a nonzero vector $v \in \mathcal{H}$, we say that v is (Q, κ)-*invariant* if $\sup_{x \in Q} \|\rho (x) \cdot v - v\| < \kappa \cdot \|v\|$. We say that (Q, κ) is a *Kazhdan pair* for Γ if every unitary representation $(\mathcal{H}, \rho)$ of Γ satisfies:

$$\text{if } \mathcal{H} \text{ contains a } (Q, \kappa)\text{-invariant vector,}$$
$$\text{then it also contains a } \Gamma\text{-invariant nonzero vector.} \tag{2.5}$$

We say that the group Γ has *Kazhdan's Property* (T) if it has a Kazhdan pair (Q, κ) for which Q is finite (and $\kappa > 0$). Every discrete group with Property (T) is finitely-generated [6,19]. If Γ has Property (T), then for every finite generating set Q of Γ, there is $\kappa > 0$ for which (Q, κ) is a Kazhdan pair for Γ, and we call such κ a *Kazhdan constant* for (Γ, Q).

Lemma 2.5. *Assume that Γ has Property* (T) *with Kazhdan constant $\kappa > 0$ for $(\Gamma, S^{\pm})$.* *Let $(\mathcal{H}_1, \alpha)$ and $(\mathcal{H}_2, \beta)$ be finite-dimensional unitary representations of Γ. Let $\epsilon > 0$, and let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a nonzero linear map, such that for each $s \in S$,*

$$\|\alpha\left(s^{-1}\right) \circ f \circ \beta\left(s\right) - f\| < \epsilon \cdot \|f\|$$

Then, there is a morphism $h : \mathcal{H}_1 \to \mathcal{H}_2$ of Γ-representations, such that

$$\|f - h\| < \frac{\epsilon}{\kappa} \cdot \|f\|$$

Proof. The map f is an (S, ϵ)-invariant vector in the representation $\mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}_1, \mathcal{H}_2\right)$ of Γ. So, there is a Γ-invariant linear map $h \in \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}_1, \mathcal{H}_2\right)$ such that $\|f - h\| < \frac{\epsilon}{\kappa} \cdot \|f\|$ (see Remark 1.1.10 of [6]). The invariance of h is equivalent to h being a morphism of Γ-representations. $\square$

We are now ready to prove the main theorem:

Proof of Theorem 1.4. Before we begin, note that for each $n \in \mathbb{N}$, the group Γ has only finitely many finite-index subgroups of index n because Γ is finitely-generated. Since, in addition, Γ has Property (T), it has only finitely many irreducible unitary representations of any given dimension $n \in \mathbb{N}$ (up to isomorphism). For the last assertion, see Theorem 2.6 of [36], or Corollary 3 of [37] for a more quantitative proof, or Proposition IV of [14] for an explicit upper bound on the number of representations. In any case, it can be proved by a simple application of Lemma 2.5, together with the compactness of $\mathrm{U}\left(n\right)^{|S|}$ and Schur's Lemma.

Let $\kappa > 0$ be a Kazhdan constant for Γ with respect to $S^{\pm}$. First, assume that Γ is $\mathcal{P}$-stable. Assume, for the sake of contradiction, that Γ has infinitely many finite-index subgroups, and let $\{\Lambda_n\}_{n=1}^{\infty}$ be a sequence of such subgroups for which $[\Gamma : \Lambda_n] \to \infty$. Fix $n \in \mathbb{N}$. Denote $\mathcal{B}_n = \Gamma / \Lambda_n = \{x_1, \ldots, x_k\}$, where $k = [\Gamma : \Lambda_n]$. Write $\alpha_n : \Gamma \to \mathrm{U}\left(\mathbb{C}\left[\mathcal{B}_n\right]\right)$ for the permutation representation produced by the action of Γ on $\mathcal{B}_n$ by multiplication from the left. Write $\mathcal{B}_n^0 = \{x_1, \ldots, x_{k-1}\}$, and let $f_n : \mathbb{C}\left[\mathcal{B}_n^0\right] \to \mathbb{C}\left[\mathcal{B}_n\right]$ be the inclusion map. By Proposition 2.3(ii), there is a function $\rho_n : S \to \mathrm{U}\left(\mathbb{C}\left[\mathcal{B}_n^0\right]\right)$, such that:

$$\forall s \in S \quad \rho_n\left(s\right) \text{ permutes } \mathcal{B}_n^0 \tag{2.6}$$

$$\forall w \in \mathrm{Ker}\left(\pi\right) \quad \|\rho_n\left(w\right) - I\|_{\mathrm{F}} \leq 3^{|w|} \tag{2.7}$$

$$\forall s \in S \quad \|\alpha_n\left(s^{-1}\right) \circ f_n \circ \rho_n\left(s\right) - f_n\|_{\mathrm{F}} \leq 2 \tag{2.8}$$

Inequality (2.7) is equivalent to

$$\forall w \in \mathrm{Ker}\left(\pi\right) \quad \|\rho_n\left(w\right) - I\|_{\mathrm{HS}} \leq \frac{3^{|w|}}{\left|\mathcal{B}_n^0\right|^{1/2}} \ . \tag{2.9}$$

From (2.6) and (2.9), we see that $(\rho_n)_{n=1}^{\infty}$ is a $\mathcal{P}$-stability-challenge for Γ. Since Γ is $\mathcal{P}$-stable, there is a solution $(\tilde{\rho}_n)_{n=1}^{\infty}$ for $(\rho_n)_{n=1}^{\infty}$, where $\tilde{\rho}_n : S \to \mathrm{U}\left(\mathbb{C}\left[\mathcal{B}_n^0\right]\right)$ and $\tilde{\rho}_n(s)$ is a permutation matrix for each $s \in S$. We may extend each $\tilde{\rho}_n : S \to \mathrm{U}\left(\mathbb{C}\left[\mathcal{B}_n^0\right]\right)$ to a representation $\tilde{\rho}_n : \Gamma \to \mathrm{U}\left(\mathbb{C}\left[\mathcal{B}_n^0\right]\right)$. Now, for each $s \in S$,

$$\frac{1}{\|f_n\|_{\mathrm{F}}} \cdot \left\| \alpha_n\left(s^{-1}\right) \circ f_n \circ \tilde{\rho}_n(s) - f_n \right\|_{\mathrm{F}}$$

$$\leq \left|\mathcal{B}_n^0\right|^{-1/2} \cdot \left\| \alpha_n\left(s^{-1}\right) \circ f_n \circ \rho_n(s) - f_n \right\|_{\mathrm{F}}$$

$$+ \left|\mathcal{B}_n^0\right|^{-1/2} \cdot \left\| \alpha_n\left(s^{-1}\right) \circ f_n \circ \left(\tilde{\rho}_n(s) - \rho_n(s)\right) \right\|_{\mathrm{F}}$$

$$\leq \left|\mathcal{B}_n^0\right|^{-1/2} \cdot 2 + \left\| \tilde{\rho}_n(s) - \rho_n(s) \right\|_{\mathrm{HS}} ,$$

where the last inequality follows from (2.8) and the fact that $\alpha_n\left(s^{-1}\right)$ and f_n are unitary. Since $(\tilde{\rho}_n)_{n=1}^{\infty}$ is a solution for $(\rho_n)_{n=1}^{\infty}$, we deduce that

$$\frac{1}{\|f_n\|_{\mathrm{F}}} \cdot \left\| \alpha_n\left(s^{-1}\right) \circ f_n \circ \tilde{\rho}_n(s) - f_n \right\|_{\mathrm{F}} \overset{n\to\infty}{\longrightarrow} 0 .$$

Thus, by Lemma 2.5, there are morphisms of representations $(h_n)_{n=1}^{\infty}$, $h_n : \mathbb{C}\left[\mathcal{B}_n^0\right] \to \mathbb{C}\left[\mathcal{B}_n\right]$, such that

$$\frac{1}{\|f_n\|_{\mathrm{F}}} \cdot \|f_n - h_n\|_{\mathrm{F}} \to 0 ,$$

in contradiction with Proposition 2.4(ii). This finishes the proof under the assumption that Γ is $\mathcal{P}$-stable.

Now, assume that Γ is HS-stable rather than $\mathcal{P}$-stable. Arguing as above, using Propositions 2.3(i) and 2.4(i) instead of Propositions 2.3(ii) and 2.4(ii), respectively, we deduce that Γ has only finitely many irreducible finite-dimensional representations up to isomorphism. Assume, for the sake of contradiction, that Γ has infinitely many subgroups of finite-index, and let $\{\Lambda_n\}_{n=1}^{\infty}$ be a sequence of such subgroups, for which $[\Gamma : \Lambda_n] \to \infty$. Write $\Lambda_0 = \Gamma$. We may assume, without loss of generality, that the subgroups $\{\Lambda_n\}_{n=1}^{\infty}$ are normal in Γ, and that $\Lambda_n \subsetneqq \Lambda_{n-1}$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Take $\gamma_n \in \Lambda_{n-1} \setminus \Lambda_n$. The regular representation $\mathbb{C}\left[\Gamma/\Lambda_n\right]$ of the finite group Γ/Λ_n is faithful, and it decomposes as a direct sum of irreducible representations of Γ. So, for at least one of these irreducible representations, call it V_n, γ_n does not act on V_n as the identity. But $\gamma_n \in \Lambda_{n-1}$, and so it acts as the identity on V_i for each $1 \leq i < n$. Therefore, we produced a sequence $\{V_n\}_{n=1}^{\infty}$ of pairwise non-isomorphic finite-dimensional irreducible representations of Γ, a contradiction. $\square$

3. The non-$\mathcal{P}$-stability of $\mathrm{Aut}(\mathbb{F}_n)$, $\mathrm{Out}(\mathbb{F}_n)$ and $\mathrm{MCG}(g)$

In this section, we focus on $\mathcal{P}$-stability, and write d_n for d_n^{Hamming}.

Theorem 3.1.

i) *For $n \geq 3$, both $\mathrm{Aut}\,(\mathbb{F}_n)$ and $\mathrm{Out}\,(\mathbb{F}_n)$ are not $\mathcal{P}$-stable.*
ii) *For $g \geq 3$, the mapping class group $\mathrm{MCG}\,(g)$ of a closed orientable surface Σ_g of genus g is not $\mathcal{P}$-stable.*

The proof uses the following proposition, which is a special case of Proposition A.3 of [5].

Proposition 3.2. *Assume that Γ is finitely-presented and $\mathcal{P}$-stable, and let N be a normal subgroup of Γ. If N is a finitely-generated group, then Γ/N is $\mathcal{P}$-stable.*

Remark 3.3. Proposition 3.2 is true also in a stronger form, where Γ is only assumed to be finitely-generated rather than finitely-presented [3]. We do not need this stronger form here.

Proof of Theorem 3.1. Let $n \geq 3$ and $g \geq 3$. The action of $\mathrm{Aut}\,(\mathbb{F}_n)$ on the abelianization of $\mathbb{F}_n$ and the action of $\mathrm{MCG}\,(g)$ on $H^1\,(\Sigma_g)$ give rise to epimorphisms:

$$\mathrm{MCG}(g) \longrightarrow \mathrm{Sp}_{2g}\,(\mathbb{Z})$$
$$\mathrm{Aut}(\mathbb{F}_n) \longrightarrow \mathrm{GL}_n\,(\mathbb{Z})$$
$$\mathrm{Out}(\mathbb{F}_n) \longrightarrow \mathrm{GL}_n\,(\mathbb{Z})$$

The groups $\mathrm{GL}_n\,(\mathbb{Z})$ and $\mathrm{Sp}_{2g}\,(\mathbb{Z})$ are infinite, residually-finite (hence sofic) and have Property (T). Hence, by Theorem 1.4, they are not $\mathcal{P}$-stable. Moreover, the kernels of all three epimorphisms are the Torelli groups, which are known to be finitely-generated (as $g \geq 3$) [18,30] (cf. [7]). Therefore, $\mathrm{Aut}(\mathbb{F}_n)$, $\mathrm{Out}(\mathbb{F}_n)$ and $\mathrm{MCG}\,(g)$ are not $\mathcal{P}$-stable by virtue of Proposition 3.2. □

4. Remarks and suggestions for further research

4.1. Stability of hyperbolic groups

It is clear that free groups are both $\mathcal{P}$-stable and HS-stable. On the other hand, lattices in the rank one simple Lie groups $\mathrm{Sp}\,(n,1)$ $(n \geq 2)$ have Property (T) (see [20] or [6]), and so they are neither $\mathcal{P}$-stable nor HS-stable by Theorem 1.3. However both free groups and the cocompact lattices among the aforementioned lattices are hyperbolic [12]. So, hyperbolicity by itself does not suffice to determine whether a group is stable. An interesting question is whether surface groups of genus $g \geq 2$ are $\mathcal{P}$-stable or HS-stable.

Proposition 3.2 provides an additional method, which does not have to use Property (T), to construct hyperbolic groups which are not $\mathcal{P}$-stable. This can be done through the Rips construction [31]: Let Λ be a finitely-presented group which is not $\mathcal{P}$-stable. For

example, take the Baumslag-Solitar group $\Lambda = \mathrm{BS}\,(2,3)$, which is not residually-finite [2], but is free-by-solvable [21], hence residually-solvable, and thus sofic. Then, the Rips construction provides a hyperbolic group Ω which admits Λ as a quotient with a finitely-generated kernel. Proposition 3.2 implies that Ω is not $\mathcal{P}$-stable. In Section 4.4, we define a more flexible notion of $\mathcal{P}$-stability. We remark that if Λ is sofic and not residually-finite, then it is not $\mathcal{P}$-stable even in the flexible sense. Hence, the same is true for the hyperbolic group Ω (as the proof of Proposition 3.2 works for flexible $\mathcal{P}$-stability as well).

4.2. Property (τ) *and Property* $(\mathrm{T};\mathrm{FD})$

The arguments presented in Section 2 do not require the full strength of Property (T) in the sense that they only go through *finite-dimensional* unitary representations of Γ. Focusing on $\mathcal{P}$-stability (rather than HS-stability), even more is true: only finite-dimensional unitary representations that *factor through finite quotients* of Γ are relevant. Recall that a discrete group Δ has Property (τ) if it has a pair (Q,κ), $|Q| < \infty$, $\kappa > 0$, such that Condition (2.5) from the definition of Property (T) holds for all finite-dimensional representations of Δ that factor through finite quotients, and it has Property (T; FD) (see [29]) if the same holds for *all* finite-dimensional representations of Δ. We get the following more general result:

Theorem 4.1. *Let* Γ *be a finitely generated group.*

  i) *If* Γ *has Property* (τ) *and is* $\mathcal{P}$-*stable, then* Γ *has only finitely many finite-index subgroups. Hence, a sofic group with Property* (τ) *is not* $\mathcal{P}$-*stable, unless it is finite.*
  ii) *If* Γ *has Property* $(\mathrm{T};\mathrm{FD})$ *and is* HS-*stable, then* Γ *has only finitely many finite-index subgroups. Hence, a hyperlinear group with Property* $(\mathrm{T};\mathrm{FD})$ *is not* HS-*stable, unless it is finite.*

Warning: The weaker notion of Property (τ) *with respect to a family of finite-index subgroups* $\{N_i\}_{i=1}^{\infty}$ does *not* suffice to deduce the conclusion of Theorem 4.1(i), even if the family is separating (i.e. $\cap N_i = \{1\}$). For example, the group $\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is free, so it is clearly $\mathcal{P}$-stable, and has Property (τ) with respect to the family of *congruence* subgroups $\{\Gamma \cap \mathrm{Ker}\,(\mathrm{SL}_2\,(\mathbb{Z}) \to \mathrm{SL}_2\,(\mathbb{Z}/m\mathbb{Z}))\}_{m=1}^{\infty}$ (it has the so called *Selberg property* [28]).

Note that it is easy to see that a free product of stable groups is stable (for all versions of stability). An interesting corollary of Theorem 4.1(i) is that a free product of two $\mathcal{P}$-stable groups, amalgamated along a finite-index subgroup, is not necessarily $\mathcal{P}$-stable. Indeed, for p an odd prime, look at

$$\Gamma\,(2) = \mathrm{Ker}\left(\mathrm{SL}_2\left(\mathbb{Z}\left[\tfrac{1}{p}\right] \right) \to \mathrm{SL}_2\left(\mathbb{Z}\left[\tfrac{1}{p}\right] / 2\mathbb{Z}\left[\tfrac{1}{p}\right] \right) \right)\,.$$

This is an amalgamated product of two free groups along a finite index subgroup (see [33], Chapter II, Section 1.4, Corollary 2), and, as with the example of $SL_2(\mathbb{Z})$ above, it has the Selberg property [28]. However, unlike $SL_2(\mathbb{Z})$, the group $SL_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$ satisfies the congruence subgroup property [32], and so from the Selberg property we deduce that it has Property (τ), and so the same is true for $\Gamma(2)$, hence the latter is not $\mathcal{P}$-stable.

4.3. Relative Property (T)

Recall that the group Γ, generated by the finite set S, has *relative Property* (T) with respect to a subgroup $N \leq \Gamma$ if there is $\kappa > 0$, such that every unitary representation $(\mathcal{H}, \rho)$ of Γ that has an (S, κ)-invariant vector $v \in \mathcal{H}$, also has an N-invariant nonzero vector. If Γ has relative Property (T) with respect to a subgroup $N \leq \Gamma$, rather than Property (T), we may deduce a weak form of Lemma 2.5, where the constructed morphism h is merely a morphism of N-representations. Using this variant of the lemma, and arguing as in the proof of Theorem 1.4, we deduce the following:

Theorem 4.2. *Assume that Γ is $\mathcal{P}$-stable and has relative Property* (T) *with respect to a subgroup $N \leq \Gamma$. Then, the collection*

$$\{L \leq \Gamma \mid [\Gamma : L] < \infty \text{ and } \Gamma = NL\}$$

is finite.

We exhibit an application of Theorem 4.2. The group $SL_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ by matrix multiplication, giving rise to a semi-direct product $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$. It is well-known that this semi-direct product has relative Property (T) with respect to the subgroup $\mathbb{Z}^2 \rtimes \{1\}$. So, the infinite collection $\left\{\left(n\mathbb{Z}^2\right) \rtimes SL_2(\mathbb{Z})\right\}_{n=1}^{\infty}$, of finite-index subgroups, exhibits the non-$\mathcal{P}$-stability of $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$. More interestingly, letting H be the finite-index subgroup of $SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, we may deduce in the same manner that $\mathbb{Z}^2 \rtimes H$ is not $\mathcal{P}$-stable as well. Note that since $\mathbb{Z}^2$ is abelian and H is free, we know that both are stable [1]! We conclude:

Corollary 4.3. *A semidirect product of finitely-presented $\mathcal{P}$-stable groups is not necessarily $\mathcal{P}$-stable.*

4.4. A flexible variant of $\mathcal{P}$-stability

Finally, let us make a remark and a suggestion for further research. Our proof of non-$\mathcal{P}$-stability of groups with Property (T) starts with a true action of Γ on a set X of n points, which is then deformed a bit into an almost action on a set of $n-1$ points. For Γ to be $\mathcal{P}$-stable, this almost action must be close to an actual action on $n-1$ points. We

proved that it is never the case if Γ has Property (T) and the action $\Gamma \curvearrowright X$ is transitive. However, the action on $n - 1$ points is clearly close to a true action on a set of n points since we started with such an action.

One may suggest a notion of "flexible $\mathcal{P}$-stability", which requires that every almost action can be corrected to an action by allowing to add points to the set before correcting it. One then needs to decide how to measure the distance between permutations of different sizes. For $\sigma \in \mathrm{Sym}\,(n)$ and $\tau \in \mathrm{Sym}\,(m)$, $n \leq m$, we generalize d_n^{Hamming} by defining

$$d_n^{\mathrm{flex}}\,(\sigma, \tau) = \frac{1}{n} \cdot \left(|\{x \in [n] \mid \sigma\,(x) \neq \tau\,(x)\}| + (m - n) \right) \ .$$

For a $\mathcal{P}$-stability-challenge $(f_k : S \to \mathrm{Sym}\,(n_k))_{k=1}^{\infty}$, we define a "flexible solution" to be a sequence of homomorphisms $(g_k : \Gamma \to \mathrm{Sym}\,(t_k))_{k=1}^{\infty}$, $t_k \geq n_k$, for which $d_{n_k}^{\mathrm{flex}}\,(f_k\,(s), g_k\,(s)) \overset{k \to \infty}{\longrightarrow} 0$ for each $s \in S$. We say that Γ is $\mathcal{P}$-*flexibly*-stable if every $\mathcal{P}$-stability-challenge for Γ has a flexible solution.

One may consider an even more flexible notion of $\mathcal{P}$-stability by replacing d_n^{flex} with

$$d_n^{\mathrm{very\text{-}flex}}\,(\sigma, \tau) = \frac{1}{n} \cdot |\{x \in [n] \mid \sigma\,(x) \neq \tau\,(x)\}| \ .$$

The flexible notions of stability suggest a path towards finding a non-sofic group: a non-residually-finite group which is $\mathcal{P}$-flexibly-stable, or even just $\mathcal{P}$-very-flexibly-stable, is non-sofic. In fact, there is a possibly more accessible path which requires less. A *sofic approximation* for Γ is a sequence $(f_k : S \to \mathrm{Sym}\,(n_k))_{k=1}^{\infty}$, $n_k \in \mathbb{N}$, such that

$$\forall w \in \mathrm{Ker}\,(\pi) \qquad d_{n_k}\left(f_k\,(w), \mathbf{1}_{G_{n_k}}\right) \overset{k \to \infty}{\longrightarrow} 0$$

$$\forall w \notin \mathrm{Ker}\,(\pi) \qquad d_{n_k}\left(f_k\,(w), \mathbf{1}_{G_{n_k}}\right) \overset{k \to \infty}{\longrightarrow} 1 \ .$$

Note that every sofic approximation is a $\mathcal{P}$-stability-challenge. Now, if a finitely-generated group Γ is not residually-finite, then Γ must be non-sofic if it satisfies the following condition: For every sofic approximation $(f_k : S \to \mathrm{Sym}\,(n_k))_{k=1}^{\infty}$ for Γ, there is a sequence of homomorphisms $(g_k : \Gamma \to \mathrm{Sym}\,(t_k))_{k=1}^{\infty}$, $t_k \geq n_k$, such that

$$\liminf_{k \to \infty} \sum_{s \in S} d_{n_k}^{\mathrm{very\text{-}flex}}\,(f_k\,(s), g_k\,(s)) = 0 \ .$$

In fact, it is enough to require that the above $\liminf$ is smaller than some small enough constant depending on $|S|$ and on the length of the shortest word exhibiting the non-residual-finiteness of Γ. This strategy generalizes the observation [11] that a non-residually-finite $\mathcal{P}$-stable group is non-sofic, and its strengthening [1] which says the same with respect to the notion of weak-stability, i.e., stability with respect to sofic approximations only, rather than general $\mathcal{P}$-stability-challenges.

One may hope that this path would make it easier to find a non-sofic group. Note that it is still an open problem whether surface groups are $\mathcal{P}$-stable, but in response to an earlier version of the present paper, Lazarovich, Levit and Minsky proved that they are $\mathcal{P}$-flexibly-stable [23].

Acknowledgments

Appendix A. Non-stability through almost-irreducible actions

In this appendix, we give an alternative proof of Theorem 1.3 under the assumption that Γ has an infinite linear quotient. In fact, we prove a stronger statement (recall that every discrete group with Property (T) is finitely generated):

Theorem A.1. *Let Γ be a discrete group with Property* (T). *Assume that Γ has an infinite linear quotient. Then Γ has a $\mathcal{P}$-stability-challenge which has no solution, even when thought of as an* HS-*stability-challenge. In particular, Γ is neither $\mathcal{P}$-stable nor* HS-*stable.*

The main ingredient in the proof of Theorem A.1 is the following result, proved below:

Theorem A.2. *Let Γ be a finitely generated group having a non-virtually-solvable infinite linear quotient. Then it has an infinite sequence of finite permutational representations $\rho_n : \Gamma \to \mathrm{Sym}\,(X_n)$ such that the corresponding unitary representations $\bar{\rho}_n : \Gamma \to \mathrm{U}\left(L^2\,(X_n)\right)$ are almost irreducible in the following sense: $\bar{\rho}_n$ has an irreducible subrepresentation χ_n such that $\frac{\dim \chi_n}{\dim \bar{\rho}_n} \overset{n \to \infty}{\longrightarrow} 1$.*

We begin by explaining how Theorem A.1 follows from Theorem A.2. Let Γ be a finitely generated group that has Property (T) and an infinite linear quotient. This linear quotient is not virtually solvable by virtue of Property (T), hence Theorem A.2 is applicable to Γ. As usual, we think of Γ as a quotient of a free group $\mathbb{F}$, generated by a finite set S. Take $\rho_n : \Gamma \to \mathrm{Sym}\,(X_n)$ and χ_n as in Theorem A.2. Choose a subset Y_n of X_n whose cardinality is $|Y_n| = \dim \chi_n$. Choose one action $\mathbb{F} \curvearrowright Y_n$ satisfying the following: for every $s \in S$ and $x \in X_n$, if s takes x to x' under the action $\mathbb{F} \curvearrowright X_n$ and both x and x' belong to Y_n, then s takes x to x' under the action $\mathbb{F} \curvearrowright Y_n$ as well. Let $\bar{\varphi}_n : S \to \mathrm{U}\left(L^2\,(Y_n)\right)$ be the unitary representation corresponding to the

action $\mathbb{F} \curvearrowright Y_n$. The sequence $(\bar{\varphi}_n)_{n=1}^{\infty}$ is an HS-stability-challenge for Γ, and can also be thought of as a $\mathcal{P}$-stability-challenge. Assume, for the sake of contradiction, that this HS-stability-challenge has a solution. Then, for large enough n, the map $\bar{\varphi}_n$ is very close to a map $\tilde{\varphi}_n : S \to \mathrm{U}\left(L^2\left(Y_n\right)\right)$ which extends to a homomorphism $\tilde{\varphi}_n : \Gamma \to \mathrm{U}\left(L^2\left(Y_n\right)\right)$. Composing the canonical inclusion $L^2\left(Y_n\right) \to L^2\left(X_n\right)$ with a projection of Γ-representations $L^2\left(X_n\right) \to \chi_n$ produces a linear map $f : L^2\left(Y_n\right) \to \chi_n$. The map f is an almost invariant non-zero vector in the Γ-representation $\mathrm{Hom}_{\mathbb{C}}\left(\left(L^2\left(Y_n\right), \tilde{\varphi}_n\right), \chi_n\right)$. Hence, since Γ has Property (T), there is a non-zero morphism of Γ-representations $\left(L^2\left(Y_n\right), \tilde{\varphi}_n\right) \to \chi_n$. This contradicts Schur's Lemma since $|Y_n| = \dim \chi_n$ and so every irreducible subrepresentation of $\left(L^2\left(Y_n\right), \tilde{\varphi}_n\right)$ is of dimension smaller or equal to $|Y_n| - 1 = \dim \chi_n - 1$.

Our proof of Theorem A.2 makes use of the following claim. Thanks are due to Bob Guralnick for useful discussions of it.

Proposition A.3. *Let $\underline{G}$ be a semisimple Chevalley group scheme and $\underline{B}$ its Borel subgroup. Write $G = \underline{G}\left(\mathbb{F}_q\right)$ and $B = \underline{B}\left(\mathbb{F}_q\right)$, where $\mathbb{F}_q$ is a finite field of order q. Then, there is an irreducible G-subrepresentation V of $L^2\left(G/B\right)$ such that $\lim_{q \to \infty} \frac{\dim V}{\dim L^2(G/B)} = 1$.*

Proof. Let $p = \mathrm{char}\,\mathbb{F}_q$. Assume first that $\underline{G}$ is a simple Chevalley group. Denote $r = \mathrm{rank}\left(\underline{G}\right)$ and write R for the number of positive roots of G. The Steinberg representation St of G is a subrepresentation of $L^2\left(G/B\right)$ (see Section 1.3 of [16]). By Theorem 6.4.7(ii) of [9], $\dim \mathrm{St}$ is equal to the cardinality of the p-Sylow subgroup S_p of G. By Lemma 54 and its corollary in [34], $|S_p| = q^R$, $|B| = q^R \cdot (q-1)^r$ and

$$|G| = q^R (q-1)^r \cdot \sum_{w \in W} q^{N(w)} \tag{A.1}$$

where the sum runs over the elements of the Weyl group W associated with $\underline{G}$, and $N(w)$ is the number of positive roots taken by w to negative roots. Exactly one element w_0 of W, the longest element, takes all positive roots to the negative ones ([17], Section 1.8). Hence, the sum in Equation (A.1) is equal to

$$q^R + \sum_{w_0 \neq w \in W} q^{N(w)}, \qquad N(w) < R\,.$$

This sum is the index of B in G, which is equal to $\dim L^2\left(G/B\right)$. Hence, $\lim_{q \to \infty} \frac{\dim \mathrm{St}}{\dim L^2(G/B)} = 1$.

Now, consider the general case where $\underline{G}$ is a semisimple Chevalley group. We can assume that $\underline{G} = \prod_{i=1}^{s} \underline{G}_i$ with $\underline{G}_i$ simple. Hence, $\underline{G}\left(\mathbb{F}_q\right)/\underline{B}\left(\mathbb{F}_q\right) = \prod_{i=1}^{s} \underline{G}_i\left(\mathbb{F}_q\right)/\underline{B}_i\left(\mathbb{F}_q\right)$ and the tensor product of the Steinberg representations of the $\underline{G}_i\left(\mathbb{F}_q\right)$ is an irreducible subrepresentation of $L^2\left(G/B\right)$ of almost full dimension by what we proved in the case of a simple Chevalley group. $\quad\square$

In the notation of the claim, the G-set G/B can be identified with the set $\mathrm{Conj}_G(B)$ of conjugates of B in G since B is self-normalizing in G.

Proof of Theorem A.2. Assume w.l.o.g. that Γ is an infinite linear group which is not virtually solvable. Dividing its Zariski closure H by the solvable radical, we can assume that H is non-trivial and semisimple (though maybe not connected). Moreover, we can assume (cf. [27], Lemma 3.6) that H^0, the identity component of H, is simply connected, and H^0 is of the form $\prod_{i=1}^s G_i$ where G_i are simple Chevalley groups.

By Section 4 of [22], we can replace Γ with a specialization of Γ, and assume that $\Gamma \subset \mathrm{GL}_n(k)$, where k is a global field, and still the Zariski closure of Γ is a k-algebraic group which is isomorphic to H over the algebraic closure $\bar{k}$ of k, and we replace H by this new Zariski closure. As Γ is finitely-generated, it lies in $H(\mathcal{O}_S)$ where $\mathcal{O}$ is the ring of integers of k and $\mathcal{O}_S$ is its localization at a finite set S of primes.

We now apply Nori-Weisfeiler strong approximation (cf. [26], Window: Strong approximation for linear groups) to $\Gamma^0 = H^0 \cap \Gamma$ to deduce that for almost every prime ideal $\mathfrak{p}$ of $\mathcal{O}_S$, Γ^0 is mapped onto $H^0(\mathcal{O}_S/\mathfrak{p})$. Moreover, applying the Chebotarev Density Theorem (as in [24], Section 4), we deduce that if G is the Chevalley (split) form of H^0, then for infinitely many $\mathfrak{p}$ (in fact, a subset of positive density), $H^0(\mathcal{O}_S/\mathfrak{p}) \cong G(\mathcal{O}_S/\mathfrak{p})$. To summarize, Γ^0 surjects onto $G = G(\mathbb{F}_q)$ for infinitely many finite fields $\mathbb{F}_q$. Fixing one such field $\mathbb{F}_q$ and letting B be the Borel subgroup of G, the action $\Gamma^0 \curvearrowright \mathrm{Conj}_G(B)$ extends to an action $\Gamma \curvearrowright \mathrm{Conj}_G(B)$ since Γ^0 is a normal subgroup of Γ. The desired conclusion now follows from Proposition A.3, the identification of G/B with $\mathrm{Conj}_G(B)$ and the fact that the largest dimension of a subrepresentation of $L^2(G/B)$ does not decrease when extending the Γ^0-representation to a Γ-representation. $\quad\square$

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